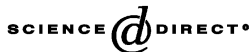




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Geometrical origin of the \ast -product in the Fedosov formalism

M. Gadella^a, M.A. del Olmo^{a,*}, J. Tosiek^b

^a *Departamento de Física Teórica, Universidad de Valladolid, E-47011 Valladolid, Spain*

^b *Institute of Physics, Technical University of Łódź, Wólczajska 219, 93-005 Łódź, Poland*

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Abstract

The construction of the \ast -product proposed by Fedosov is implemented in terms of the theory of fibre bundles. The geometrical origin of the Weyl algebra and the Weyl bundle is shown. Several properties of the product in the Weyl algebra are proved. Symplectic and abelian connections in the Weyl algebra bundle are introduced. Relations between them and the symplectic connection on a phase space M are established. Elements of differential symplectic geometry are included. Examples of the Fedosov formalism in quantum mechanics are given.

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1. Introduction

The standard formulation of quantum mechanics in terms of a complex Hilbert space and linear operators is mainly applied for systems, whose classical limit may be de-

* Corresponding author. Tel.: +34 983423146; fax: +34 983423013.

E-mail addresses: gadella@fta.uva.es (M. Gadella); olmo@fta.uva.es (M.A. del Olmo); tosiek@p.lodz.pl (J. Tosiek).

scribed on a phase space \mathbb{R}^{2n} . There are two reasons for this situation: the fundamental operators $\hat{q}^i, \hat{p}_i, i = 1, \dots, 2n$, are well defined by the Dirac quantization scheme [1] only in the case of \mathbb{R}^{2n} , and the operator orderings are based on the Fourier transform [2–4].

Unless we agree to weaken the foregoing assumptions [5] to deal with other quantum systems it is necessary to use geometric quantization [6,7] or deformation quantization.

Classical mechanics is a physical theory which works perfectly on arbitrary differential manifolds. From this reason shortly after presenting a standard version of quantum theory, researchers began to look for an equivalent formulation of quantum mechanics based on differential geometry. The first complete version of quantum theory in the language of the theory of manifolds appeared in the middle of the XXth century, when Moyal [8] using previous works by Weyl [9], Wigner [10] and Groenewold [11] presented quantum mechanics as a statistical theory. His results are only valid for the case \mathbb{R}^{2n} . However, the paper by Moyal contains the seminal ideas about deformation quantization, since the main result of this work is the substitution of the point-wise product of functions in phase space for a new product which is a formal power series in \hbar .

A modern version of Moyal’s deformation quantization on an arbitrary differential manifold was proposed by Bayen et al. [12] in 1978. The mathematical structure of this formulation of the quantum theory is based, like Hamiltonian classical mechanics, on differential geometry of symplectic spaces. Observables are smooth real functions on a phase space and states are represented by functionals. Macroworld appears in this formalism as the limit of the quantum reality for the Planck constant \hbar tending to 0^+ .

The list of axioms constituting deformation quantization looks as follows:

- (i) a state of a physical system is described on a $2n$ -dimensional phase space M ,
- (ii) an observable is a real smooth function on M ,
- (iii) for every complex-valued smooth functions (f, g, h) of $C^\infty(M)$ the $*$ -product fulfills the following conditions:

$$(a) \quad f * g = \sum_{t=0}^{\infty} \left(\frac{i\hbar}{2}\right)^t M_t(f, g),$$

where $M_t(\cdot, \cdot)$ is a bidifferential operator on M (see definition later, formula (1.1)).

- (b) The element M_0 represents the ‘usual’ commutative product of functions i.e.

$$M_0(f, g) = f \cdot g.$$

Thus, at the classical limit

$$\lim_{\hbar \rightarrow 0^+} f * g = f \cdot g.$$

- (c) The quasi-Dirac quantization postulate holds

$$M_1(f, g) - M_1(g, f) = 2\{f, g\}_P,$$

where $\{\cdot, \cdot\}_P$ stands up for Poisson brackets.

- (d) Associativity also holds

$$\sum_{t+u=s} (M_t(M_u(f, g), h) - M_t(f, M_u(g, h))) = 0 \quad \forall s \geq 0.$$

(e) For the constant function equal to 1, we have

$$M_t(1, f) = M_t(f, 1) = 0 \quad \forall t \geq 1.$$

Let us comment the postulates written above. Assumptions (i) and (ii) say that the mathematical structure of deformation quantization is modelled after the classical Hamiltonian mechanics. Differences between classical and quantum descriptions appear at the level of product of functions and representation of states. In the quantum case the product must be noncommutative. The relation between this $*$ -product and the usual point-wise multiplication of functions is established in (iii-b). The classical Lie algebra of functions is determined by the term M_1 from (iii-a). Associativity (iii-d) is analogous to the associativity in an algebra of linear operators in traditional quantum mechanics. Linearity of the bidifferential operators M_t plus postulate (iii-e) expresses the fact that the measurement of a constant quantity does not interact with any other measurement.

Let us remind what is a bidifferential operator. Let M be a n -dimensional differential manifold and (U, φ) a local chart on it with coordinates $\mathbf{q} = (q^1, \dots, q^n)$. By a bidifferential operator, $M_t(\cdot, \cdot) : C^\infty(M) \times C^\infty(M) \rightarrow \mathbb{C}$, we understand the map which locally in a chart (U, φ) is a finite linear combination of terms like

$$l(\mathbf{q}) \frac{\partial^m f(\mathbf{q})}{(\partial q^1)^{\alpha_1} \dots (\partial q^n)^{\alpha_n}} \cdot \frac{\partial^r g(\mathbf{q})}{(\partial q^1)^{\beta_1} \dots (\partial q^n)^{\beta_n}}, \quad l, f, g \in C^\infty(M), \tag{1.1}$$

where $\alpha_1 + \dots + \alpha_n = m, \beta_1 + \dots + \beta_n = r$.

Note that the assumptions of deformation quantization do not give a construction of the $*$ -product. In the simplest case when the phase space is \mathbb{R}^{2n} , there exists a relation W between a wide class of functions \mathcal{O} defined on \mathbb{R}^{2n} and linear operators $\hat{\mathcal{O}} \subset \mathcal{L}(\mathcal{H})$ acting on a Hilbert space \mathcal{H} . More information about the class of operators and functions, for which such a relation exists, can be found in [13]. The W mapping, known as the Weyl application, in the two-dimensional case takes the form [14]

$$\hat{F} = W(F) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \alpha(\hbar\lambda\mu) \exp(i[\lambda\hat{p} + \mu\hat{q}]) \tilde{F}(\lambda, \mu) d\lambda d\mu, \tag{1.2}$$

where \hat{F} is a linear operator acting in \mathcal{H} , $\alpha(\hbar\lambda\mu)$ is a function characterizing ordering of operators (however, in this work we will consider the case of Weyl ordering for which $\alpha(\hbar\lambda\mu) = 1$), p and q are canonically conjugate coordinates on the phase space \mathbb{R}^2 , \hat{p} and \hat{q} are self-adjoint operators representing momentum and position, respectively, and fulfilling the commutation relation

$$[\hat{q}, \hat{p}] = i\hbar\mathbf{1}.$$

Finally, $\tilde{F}(\lambda, \mu)$ is the Fourier transform of the function $F(p, q)$ defined by

$$\tilde{F}(\lambda, \mu) := \int_{\mathbb{R}^2} F(p, q) \exp(-i[\lambda p + \mu q]) dp dq.$$

The generalization of expression (1.2) to the case \mathbb{R}^{2n} is straightforward.

Considering the $*$ -product as the image of the product of operators by the inverse mapping W^{-1} of the Weyl application W (with $\alpha(\hbar\lambda\mu) = 1$) we obtain that

$$F * G := W^{-1}(\hat{F} \cdot \hat{G}) = F \exp\left(\frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}}\right) G, \tag{1.3}$$

where $\overleftrightarrow{\mathcal{P}}$ is the Poisson operator

$$\overleftrightarrow{\mathcal{P}} := \frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q}.$$

The arrows indicate the acting direction of the partial differential derivatives. Explicitly

$$F * G = \sum_{r=0}^{\infty} \sum_{l=0}^r \frac{1}{r!} \left(\frac{i\hbar}{2}\right)^r (-1)^l \frac{\partial^r F}{\partial q^{r-l} \partial p^l} \frac{\partial^r G}{\partial p^{r-l} \partial q^l}. \tag{1.4}$$

The mapping W^{-1} is called the Weyl correspondence and it constitutes an isomorphism between some algebra of linear operators and an algebra of functions on \mathbb{R}^{2n} . From this fact we deduce that the $*$ -product defined as (1.3) is associative. Expressions (1.3) and (1.4) are true only in coordinate systems in which a symplectic form ω on \mathbb{R}^2 is canonical and symplectic connection vanishes.

The construction presented above cannot be used in the general case because on an arbitrary manifold we are not able to introduce the Fourier transform, which is fundamental for defining the Weyl correspondence W^{-1} and the Weyl application W .

As we mentioned before, Ref. [12] does not contain an explicit procedure for obtaining a $*$ -product, only theorems of existence of $*$ -products for symplectic manifolds are presented. However, other aspects of this paper have had a great influence in the modern development of deformation quantization. For instance, the deformation of the algebraic structure of the classical Poisson algebra $C^\infty(M)$ via the $*$ -product provides an example of algebra deformation in the sense of Gerstenhaber [15]. Later, in the eighties, the algebraic deformation à la Gerstenhaber also appeared in relation to quantum groups [16]. Different aspects about deformation quantization and its applications can be found in [17–62]. For those who are interested in general aspects of quantization deformation we recommend the reviews [63–66].

The practical realization of deformation quantization on a symplectic manifold was proposed by Fedosov [67,68]. His construction is purely geometric and it is based on the theory of fibre bundles [69]. Fedosov starts from some symplectic manifold endowed with a symmetric connection and lifts this connection to the so called Weyl algebra bundle. Next he builds some new flat connection in the Weyl bundle and shows how to operate on flat sections of the Weyl algebra bundle in order to define a $*$ -product. In [70–75] one can find several considerations devoted to several aspects of the Fedosov formalism.

The aim of this paper is to present the Fedosov theory as an example of application of the theory of fibre bundles in physics. Thus, we analyze the construction of the $*$ -product, paying attention to the geometrical aspects using the formalism of fibre bundles. All the final results presented here are known but we have obtained them applying consequently the fibre bundle methods and in many cases the proofs are new and easier than the original ones.

This fact adds a pedagogical value to this work which can complement the monographies devoted to the Fedosov theory. As we mention before, we are mainly interested in the geometrical aspects of the Fedosov construction. For that reason we omit some long and technical proofs not connected directly with the geometrical nature of the topic.

The paper is organized as follows. In Section 2 we present some results about symplectic geometry that we will use later. Next sections are devoted to the main mathematical structures involved on the Fedosov method. In Section 3, starting from the construction of the Weyl algebra we obtain the Weyl bundle. This algebra is equipped with a symplectic connection, which is studied in detail in Section 4. Making the symplectic connection flat and defining a one-to-one mapping between formal series $\Pi(\hbar, C^\infty(M))$ and flat sections of the Weyl bundle, we construct in Section 5 a noncommutative associative $*$ -product on M . Some examples are presented in Section 6.

2. Symplectic geometry

This part is devoted to review some aspects of the geometry of symplectic spaces. We present a procedure for defining parallel transport for such spaces and analyze the similarities and differences between Riemannian and symplectic geometry. The reader interested in the mathematical aspects of symplectic geometry can find a systematic presentation of this topic in [76,77]. Physical applications of the symplectic geometry are analyzed in [78–81].

A symplectic manifold (M, ω) is a manifold M equipped with a nondegenerate closed 2-form ω , which is called symplectic form. The dimension of (M, ω) , which is always even, will be denoted by $\dim M = 2n$.

Let (U_i, φ_i) be some local chart on (M, ω) . In this chart the symplectic form may be written as

$$\omega = \omega_{ij} dq^i \wedge dq^j, \quad i, j = 1, 2, \dots, 2n.$$

The well known Darboux theorem establishes that in the neighborhood of each point \mathbf{p} of a symplectic manifold there exist local coordinates (q^1, \dots, q^{2n}) , called canonical or Darboux coordinates, such that the form ω may be written by means of these coordinates as

$$\omega = dq^{n+1} \wedge dq^1 + dq^{n+2} \wedge dq^2 + \dots + dq^{2n} \wedge dq^n. \quad (2.1)$$

An atlas $\{(U_\varrho, \varphi_\varrho)\}_{\varrho \in I}$ consisting of Darboux charts is called Darboux atlas.

As the form ω is nondegenerate, it establishes an isomorphism I between tangent $T_{\mathbf{p}}M$ and cotangent $T_{\mathbf{p}}^*M$ spaces at each point $\mathbf{p} \in M$, which is defined by:

$$I_X(Y) = \omega(Y, X) \quad \forall Y \in T_{\mathbf{p}}M,$$

where the 1-form $I_X \in T_{\mathbf{p}}^*M$ is the image of $X \in T_{\mathbf{p}}M$ by I .

In a local chart (U_i, φ_i) on (M, ω) in natural bases $\{dq^1, \dots, dq^{2n}\}$ and $\{(\partial/\partial q^1), \dots, (\partial/\partial q^{2n})\}$ of T_p^*M and T_pM , respectively, we can write that

$$(I_X)_r = \omega_{ir} X^i.$$

The inverse mapping $I^{-1} : T_p^*M \rightarrow T_pM$ can be obtained by

$$X^r = \omega^{ri} (I_X)_i, \tag{2.2}$$

where ω^{ri} is a covariant tensor for which the following relation holds

$$\omega^{ij} \omega_{jk} = \delta_k^i. \tag{2.3}$$

The Poisson bracket of two functions, $f, g \in C^\infty(M)$, is defined as

$$\{f, g\} = -\omega(I^{-1}(df), I^{-1}(dg)).$$

In local coordinates

$$\{f, g\} = \omega^{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j}.$$

In fact, it is possible to define a Poisson structure on a manifold without introducing a symplectic form. In such cases by a Poisson structure on a manifold M we understand an antisymmetric bilinear mapping $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ fulfilling the Jacobi identity

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$$

and the Leibniz rule

$$\{f, gh\} = g\{f, h\} + \{f, g\}h.$$

Poisson manifolds, $(M, \{\cdot, \cdot\})$, are natural generalizations of symplectic manifolds [79].

The deformation quantization programme works on Poisson manifolds [51,59,73], but we are going to continue our considerations for the case when the phase space of a system is some symplectic manifold.

Definition 2.1. A torsion-free connection ∇ in the tangent bundle TM is called symplectic if the induced connection ∇ in $T^*M \otimes T^*M$ fulfills the condition

$$\nabla \omega = 0. \tag{2.4}$$

In a local chart (U_i, φ_i) on M for $\dim M = 2n$ the requirement (2.4) is equivalent to the system of $2n^2(2n - 1)$ independent equations

$$\omega_{ij;k} := \frac{\partial \omega_{ij}}{\partial q^k} - \Gamma_{ik}^l \omega_{lj} - \Gamma_{jk}^l \omega_{il} = 0, \tag{2.5}$$

where Γ_{ik}^l are coefficients of the connection ∇ . They are symmetric in lower indices because the connection is torsion-free. In a Darboux chart all the partial derivatives $\partial\omega_{ij}/\partial q^k$ vanish, so the equation system (2.5) is equivalent to the following one

$$-\Gamma_{jik} + \Gamma_{ijk} = 0,$$

where

$$\Gamma_{ijk} := \omega_{li} \Gamma_{jk}^l.$$

Corollary 2.1. *A connection ∇ on the symplectic manifold (M, ω) is symplectic iff in every Darboux chart the coefficients Γ_{ijk} are symmetric in all the indices $\{i, j, k\}$.*

Unlike the Riemannian geometry on a symplectic manifold (M, ω) we can define many symplectic connections. The following statement holds.

Theorem 2.1 (Fedosov [68]). *The symplectic connection on a symplectic manifold (M, ω) is not unique. The set of coefficients*

$$\tilde{\Gamma}_{ijk} := \Delta_{ijk} + \Gamma_{ijk}, \quad 1 \leq i, j, k \leq \dim M,$$

where Δ_{ijk} denotes the coefficients of a tensor symmetric with respect to indices $\{i, j, k\}$ and Γ_{ijk} the coefficients of a symplectic connection on (M, ω) , determines a symplectic connection on (M, ω) .

Note that locally symplectic connections exist on any symplectic manifold. The construction of a symplectic connection on the whole manifold (M, ω) can be done using a C^∞ -partition of unity.

Definition 2.2 (Curtis and Miller [82]). A manifold M admits a C^∞ -partition of unity if, given a locally finite open cover $\{U_i\}_{i \in I}$, there exists a family of C^∞ -maps $\psi_i : M \rightarrow [0, 1]$ such that the set of points at which ψ_i does not vanish, $\text{supp}(\psi_i)$, is contained in U_i , i.e. $\text{supp}(\psi_i) \subset U_i$, and $\sum_{i \in I} \psi_i(\mathbf{p}) = 1$ for all $\mathbf{p} \in M$.

Theorem 2.2 (Curtis and Miller [82]). *Every manifold M admits a C^∞ -partition of unity.*

Since Theorem 2.2 holds, on an arbitrary symplectic manifold (M, ω) we build the symplectic connection on (M, ω) following the algorithm:

1. We cover the manifold (M, ω) with a Darboux atlas $\{(U_i, \varphi_i)\}_{i \in I}$. According to Theorem 2.2 there exists a C^∞ -partition of unity $\{\psi_i\}_{i \in I}$ compatible with $\{U_i\}_{i \in I}$.
2. In each chart (U_i, φ_i) we define a symplectic connection by giving coefficients $(\Gamma_{jkl})_i$ symmetric in all the indices.

- The coefficients of the symplectic connection ∇ on (M, ω) at an arbitrary fixed point $\mathbf{p} \in M$ are expressed by

$$\Gamma_{jkl}(\mathbf{p}) := \sum_{i \in I} \psi_i(\mathbf{p})(\Gamma_{jkl})_i.$$

Definition 2.3. Let (M, ω) be a symplectic manifold with a symplectic connection ∇ . The triad (M, ω, ∇) is called Fedosov manifold.

The curvature of a symplectic connection ∇ is characterized by a curvature tensor.

Definition 2.4. The curvature tensor R of a symplectic connection ∇ in the tangent bundle TM is a mapping $R : TM \times TM \times TM \rightarrow TM$ fulfilling the relation

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \quad \forall X, Y, Z \in TM. \tag{2.6}$$

In the natural basis $\{(\partial/\partial q^1), \dots, (\partial/\partial q^{2n})\}$ of TM the components of the curvature tensor are expressed as

$$R \left(\frac{\partial}{\partial q^j}, \frac{\partial}{\partial q^k} \right) \frac{\partial}{\partial q^i} = R^m_{ijk} \frac{\partial}{\partial q^m}.$$

In terms of the Christoffel symbols

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ki}}{\partial q^j} - \frac{\partial \Gamma^l_{ij}}{\partial q^k} + \Gamma^m_{ki} \Gamma^l_{jm} - \Gamma^m_{ij} \Gamma^l_{km}. \tag{2.7}$$

Note that for the symplectic connections of a symplectic manifold we may have different curvature tensors. For instance, the same symplectic manifold can be equipped with flat (all the components R^l_{ijk} of curvature tensor vanish) or nonvanishing curvature.

Theorem 2.3. Let us consider two Fedosov manifolds (M, ω, ∇) and $(M, \omega, \tilde{\nabla})$ for which the relation holds $\tilde{\nabla} = \nabla + \Delta$ (see Theorem 2.1). The curvature tensors R and \tilde{R} fulfill the equation

$$\tilde{R} = R + r(\Delta),$$

where $r(\Delta)$ is some tensor depending only on Δ .

Lowering the upper index in R^l_{ijk} we define a new tensor

$$R_{ijkl} := \omega_{mi} R^m_{jkl} \tag{2.8}$$

with the following properties [83,84]:

- The curvature tensor is antisymmetric in the last two indices

$$R_{ijkl} = -R_{ijlk}. \tag{2.9}$$

Indeed, from (2.7) and the fact that the connection ∇ is symmetric we can see that $R^l_{ijk} = -R^l_{ikj}$. Making the contraction (2.8) we obtain (2.9).

2. The first Bianchi identity holds

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0. \tag{2.10}$$

Effectively, this identity is equivalent to the relation

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0.$$

It means that

$$R\left(\frac{\partial}{\partial q^k}, \frac{\partial}{\partial q^l}\right) \frac{\partial}{\partial q^j} + R\left(\frac{\partial}{\partial q^l}, \frac{\partial}{\partial q^j}\right) \frac{\partial}{\partial q^k} + R\left(\frac{\partial}{\partial q^j}, \frac{\partial}{\partial q^k}\right) \frac{\partial}{\partial q^l} = 0. \tag{2.11}$$

On the other hand, the Lie brackets

$$\left[\frac{\partial}{\partial q^j}, \frac{\partial}{\partial q^k}\right] = 0, \quad j, k = 1, \dots, 2n. \tag{2.12}$$

Moreover, the connection ∇ is symmetric, so that

$$\nabla_{\partial/\partial q^k} \frac{\partial}{\partial q^j} = \nabla_{\partial/\partial q^j} \frac{\partial}{\partial q^k}, \quad j, k = 1, \dots, 2n. \tag{2.13}$$

From the definition of the curvature (2.6), involving (2.12) we rewrite the l.h.s. of (2.11) in the form

$$\begin{aligned} &\nabla_{\partial/\partial q^k} \nabla_{\partial/\partial q^l} \frac{\partial}{\partial q^j} - \nabla_{\partial/\partial q^l} \nabla_{\partial/\partial q^k} \frac{\partial}{\partial q^j} + \nabla_{\partial/\partial q^l} \nabla_{\partial/\partial q^j} \frac{\partial}{\partial q^k} - \nabla_{\partial/\partial q^j} \nabla_{\partial/\partial q^l} \frac{\partial}{\partial q^k} \\ &+ \nabla_{\partial/\partial q^j} \nabla_{\partial/\partial q^k} \frac{\partial}{\partial q^l} - \nabla_{\partial/\partial q^k} \nabla_{\partial/\partial q^j} \frac{\partial}{\partial q^l}. \end{aligned}$$

From (2.13) we can see that the foregoing expression vanishes.

3. The second Bianchi identity is verified, i.e.

$$R_{mjkl;i} + R_{mjik;l} + R_{mjli;k} = 0.$$

Indeed, this identity is equivalent to the relation

$$R^m_{jkl;i} + R^m_{jik;l} + R^m_{jli;k} = 0. \tag{2.14}$$

It is always possible to consider locally such a chart on (M, ω, ∇) that at an arbitrary fixed point $\mathbf{p} \in M$ all the coefficients Γ^i_{jk} disappear. If we compute all the covariant derivatives from (2.14) at \mathbf{p} , we obtain that the l.h.s. of (2.14) equals

$$\frac{\partial^2 \Gamma^m_{lj}}{\partial q^i \partial q^k} - \frac{\partial^2 \Gamma^m_{kj}}{\partial q^i \partial q^l} + \frac{\partial^2 \Gamma^m_{kj}}{\partial q^l \partial q^i} - \frac{\partial^2 \Gamma^m_{ji}}{\partial q^l \partial q^k} + \frac{\partial^2 \Gamma^m_{ij}}{\partial q^k \partial q^l} - \frac{\partial^2 \Gamma^m_{lj}}{\partial q^k \partial q^i} = 0.$$

It is easy to see that in fact properties 1–3 hold for any curvature tensor of a symmetric connection. Now we are going to present some features of the symplectic curvature tensors exclusively [83].

4. The symplectic curvature tensor R_{ijkl} is symmetric in the first two indices, i.e.

$$R_{ijkl} = R_{jikl}. \tag{2.15}$$

The proof is as follows. From the Darboux theorem we can always find a system of coordinates in which the symplectic tensor ω_{ij} is locally constant. In such a chart we have

$$R_{ijkl} = \frac{\partial \Gamma_{ilj}}{\partial q^k} - \frac{\partial \Gamma_{ijk}}{\partial q^l} + \omega^{zu} \Gamma_{ulj} \Gamma_{ikz} - \omega^{zu} \Gamma_{ujk} \Gamma_{ilz}, \tag{2.16}$$

$$R_{jikl} = \frac{\partial \Gamma_{jli}}{\partial q^k} - \frac{\partial \Gamma_{jik}}{\partial q^l} + \omega^{zu} \Gamma_{uli} \Gamma_{jkz} - \omega^{zu} \Gamma_{uik} \Gamma_{jlz}. \tag{2.17}$$

Permuting repeated indices $z \leftrightarrow u$ in (2.17) we see that

$$R_{jikl} = \frac{\partial \Gamma_{jli}}{\partial q^k} - \frac{\partial \Gamma_{jik}}{\partial q^l} + \omega^{uz} \Gamma_{zli} \Gamma_{jku} - \omega^{uz} \Gamma_{zik} \Gamma_{jlu}.$$

Remembering that $\omega^{zu} = -\omega^{uz}$ and using the fact that in Darboux coordinates the coefficients Γ_{jli} are symmetric in all the indices we obtain that $R_{ijkl} = R_{jikl}$.

5. For a symplectic curvature tensor the following relation holds

$$R_{ijkl} + R_{jkli} + R_{klij} + R_{lijk} = 0. \tag{2.18}$$

Indeed, from the first Bianchi identity (2.10) we get

$$\begin{aligned} R_{ijkl} + R_{iljk} + R_{iklj} &= 0, & R_{jikl} + R_{jlik} + R_{jkli} &= 0, \\ R_{kijl} + R_{kjli} + R_{klij} &= 0, & R_{lijk} + R_{lki j} + R_{ljki} &= 0. \end{aligned}$$

From (2.9) and (2.15) adding formulas (2.19) we conclude that Eq. (2.16) is always true for any symplectic curvature.

Let us compute the number of independent components of the tensor R_{ijkl} . As usually, we assume that $\dim M = 2n$. From the properties of antisymmetry (2.9) and symmetry (2.15) we obtain that the tensor R_{ijkl} has $n^2(2n + 1)(2n - 1)$ independent elements. The first Bianchi identity (2.10) is a new constraint if and only if all the indices j, k, l are different in R_{ijkl} . It means that (2.10) provides us

$$2n \binom{2n}{3}$$

new equations. The property (2.18) can now be reduced by the symmetry of the tensor or by the first Bianchi identity only if all the indices i, j, k, l are different, so we have

$$\binom{2n}{4}$$

independent conditions. Hence, the symplectic curvature tensor R_{ijkl} has

$$n(2n - 1)(n(2n + 1) - \frac{2}{3}n(2n - 2) - \frac{1}{12}(2n - 2)(2n - 3))$$

independent components. For example, on a two-dimensional Fedosov manifold (M, ω, ∇) its curvature tensor has three independent components. For $\dim M = 4$ the amount of independent elements of R_{ijkl} increases to 47. When M is a Riemannian manifold, the number of independent components of the curvature tensor would be equal to 1 and 20, respectively.

Apart from the curvature tensor R_{ijkl} the geometry of a symplectic space can be characterized by a Ricci tensor and a scalar of curvature.

Definition 2.5. The Ricci tensor on a Fedosov manifold (M, ω, ∇) is defined by

$$K_{ij} := \omega^{kl} R_{likj} = R_{ikj}^k.$$

The Ricci tensor is symmetric, i.e. $K_{ij} = K_{ji}$.

Corollary 2.2. On any Fedosov manifold

$$\omega^{ji} K_{ij} = 0. \tag{2.19}$$

This conclusion is a straightforward consequence of the fact that the Ricci tensor K_{ij} is symmetric and ω^{ji} antisymmetric. From (2.19) we can see that the scalar of any symplectic curvature defined as $K := \omega^{ji} K_{ij}$ is always 0.

3. The Weyl bundle

In this section we present the construction of a bundle, which plays a fundamental role in the definition of the $*$ -product on a symplectic manifold. Let us start defining a formal series over a vector space.

Definition 3.1 (Bordemann et al. [71]). Let λ be a fixed real number and V some vector space. A formal series in the formal parameter λ is each expression of the form

$$v[[\lambda]] = \sum_{i=0}^{\infty} \lambda^i v_i, \quad v_i \in V.$$

The set of formal series $v[[\lambda]]$ constitutes a vector space. Addition and multiplication by a scalar $a \in \mathbb{C}$ are defined, respectively, by

$$u[[\lambda]] + v[[\lambda]] = \sum_{i=0}^{\infty} \lambda^i (u_i + v_i), \quad a \cdot v[[\lambda]] = \sum_{i=0}^{\infty} \lambda^i (av_i).$$

The vector space of formal series over the vector space V in the parameter λ will be denoted by $V[[\lambda]]$ and can be considered as a direct sum

$$V[[\lambda]] = \bigoplus_{i=1}^{\infty} V_i, \quad V_i = V \forall i.$$

Let (M, ω) be a symplectic manifold and $T_{\mathbf{p}}^*M$ the cotangent space over \mathbb{C} to M at the point \mathbf{p} of M . The space $(T_{\mathbf{p}}^*M)^l$ is a symmetrized tensor product of $T_{\mathbf{p}}^*M \odot^{1\text{-times}} \dots \odot T_{\mathbf{p}}^*M$. It is spanned by

$$v_{i_1} \odot \dots \odot v_{i_l} := \frac{1}{l!} \sum_{\text{all permutations}} v_{\sigma i_1} \otimes \dots \otimes v_{\sigma i_l},$$

where $v_{i_1}, \dots, v_{i_l} \in T_{\mathbf{p}}^*M$.

Definition 3.2. A preweyl vector space $P_{\mathbf{p}}^*M$ at the point $\mathbf{p} \in M$ is the direct sum

$$P_{\mathbf{p}}^*M := \bigoplus_{l=0}^{\infty} (T_{\mathbf{p}}^*M)^l.$$

We introduce the formal series over the preweyl vector space as follows.

Definition 3.3. A Weyl vector space $P_{\mathbf{p}}^*M[[\hbar]]$ is the vector space of formal series over the preweyl vector space $P_{\mathbf{p}}^*M$ in the formal parameter \hbar .

For further physical applications we usually identify \hbar with the Planck constant. The elements of $P_{\mathbf{p}}^*M[[\hbar]]$ can be written in the form

$$v_{\mathbf{p}}[[\hbar]] = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^k a_{k,i_1,\dots,i_l}, \quad a_{k,i_1,\dots,i_l} \in P_{\mathbf{p}}^*M.$$

For $l = 0$ we have just the sum $\sum_{k=0}^{\infty} \hbar^k a_k$.

Definition 3.4 (Fedosov [68]). The degree, $\text{deg } a_{k,i_1,\dots,i_l}$, of a vector a_{k,i_1,\dots,i_l} of the Weyl vector space $P_{\mathbf{p}}^*M[[\hbar]]$ equals $2k + l$.

Our aim is to define a product (denoted by \circ) of elements of the Weyl space $P_{\mathbf{p}}^*M[[\hbar]]$ which equips the Weyl space with an algebra structure. Such a product must give a symmetric tensor. Moreover, we require that $\deg(a \circ b) = \deg a + \deg b$.

Let us assume that the elements of the Weyl vector space are written in a natural basis constructed in terms of symmetric tensor products of $\{dq^1, \dots, dq^{2n}\}$, where $\dim M = 2n$, and $X_{\mathbf{p}} \in T_{\mathbf{p}}M$ is some fixed vector of the space $T_{\mathbf{p}}M$ tangent to M at the point \mathbf{p} . Let us denote the components of X in the basis $\{(\partial/\partial q^1), \dots, (\partial/\partial q^{2n})\}$ by $X_{\mathbf{p}}^i$.

It is obvious that for every $a_{k,i_1,\dots,i_l} \in P_{\mathbf{p}}^*M[[\hbar]]$

$$a_{k,i_1,\dots,i_l}(\underbrace{X_{\mathbf{p}}, \dots, X_{\mathbf{p}}}_{l\text{-times}}) = a_{k,i_1,\dots,i_l} X_{\mathbf{p}}^{i_1} \cdots X_{\mathbf{p}}^{i_l}$$

is a complex number and we can handle $a_{k,i_1,\dots,i_l}(X_{\mathbf{p}}, \dots, X_{\mathbf{p}})$ as a polynomial of l th degree in the components of the vector $X_{\mathbf{p}}$. We extend this observation to each element of the Weyl algebra $P_{\mathbf{p}}^*M[[\hbar]]$ and consider $v_{\mathbf{p}}[[\hbar]](X_{\mathbf{p}})$ like a function of $X_{\mathbf{p}}^1, \dots, X_{\mathbf{p}}^{2n}$ of the form

$$v_{\mathbf{p}}[[\hbar]](X_{\mathbf{p}}) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^k a_{k,i_1,\dots,i_l} X_{\mathbf{p}}^{i_1} \cdots X_{\mathbf{p}}^{i_l}. \tag{3.1}$$

We do not define any topology in the space of functions of the kind (3.1).

By $\partial v_{\mathbf{p}}[[\hbar]]/\partial X_{\mathbf{p}}^i$ we understand the derivative of the sum (3.1) as a function of $X_{\mathbf{p}}^1, \dots, X_{\mathbf{p}}^{2n}$.

Definition 3.5 (Fedosov [67]). The product $\circ : P_{\mathbf{p}}^*M[[\hbar]] \times P_{\mathbf{p}}^*M[[\hbar]] \rightarrow P_{\mathbf{p}}^*M[[\hbar]]$ of two elements $a, b \in P_{\mathbf{p}}^*M[[\hbar]]$ is an element $c \in P_{\mathbf{p}}^*M[[\hbar]]$ such that for each $X_{\mathbf{p}} \in T_{\mathbf{p}}M$ the following equality holds

$$\begin{aligned} c(X_{\mathbf{p}}) &= a(X_{\mathbf{p}}) \circ b(X_{\mathbf{p}}) : \\ &= \sum_{t=0}^{\infty} \frac{1}{t!} \left(\frac{i\hbar}{2}\right)^t \omega^{i_1 j_1} \dots \omega^{i_t j_t} \frac{\partial^t a(X_{\mathbf{p}})}{\partial X_{\mathbf{p}}^{i_1} \cdots \partial X_{\mathbf{p}}^{i_t}} \frac{\partial^t b(X_{\mathbf{p}})}{\partial X_{\mathbf{p}}^{j_1} \cdots \partial X_{\mathbf{p}}^{j_t}}. \end{aligned} \tag{3.2}$$

The pair $(P_{\mathbf{p}}^*M[[\hbar]], \circ)$ is a noncommutative associative algebra called the Weyl algebra and denoted by $\mathcal{P}^*M_{\mathbf{p}}[[\hbar]]$. Let us simply enumerate some properties of the \circ -product:

1. The \circ -product is independent of the chart. This result is a straightforward consequence of the fact that all the elements appearing on the r.h.s. of the formula (3.2) are scalars.
2. The \circ -product is associative but in general nonabelian. The associativity of the \circ -product may be roughly explained in the following way. The elements $a(X_{\mathbf{p}})$ and $b(X_{\mathbf{p}})$ are polynomials in X^i 's. Let us formally substitute $X^i \rightarrow q^i$, where by q^i we denote the Cartesian coordinates on the symplectic space $(\mathbb{R}^{2n}, \omega)$. After this substitution the product for polynomials in variables q^i 's defined like (3.2) is a associative Moyal product (see the relation (1.3)). Indeed, in Darboux coordinates in the two-dimensional case with $\omega = dq^2 \wedge dq^1$,

the formula (3.2) may be written as

$$a \circ b = \sum_{t=0}^{\infty} \frac{1}{t!} \left(\frac{i\hbar}{2}\right)^t \sum_{l=0}^t (-1)^l \frac{\partial^t a(X_{\mathbf{p}})}{\partial(X_{\mathbf{p}}^1)^{t-l} \partial(X_{\mathbf{p}}^2)^l} \frac{\partial^t b(X_{\mathbf{p}})}{\partial(X_{\mathbf{p}}^1)^l \partial(X_{\mathbf{p}}^2)^{t-l}},$$

which is exactly (1.4).

3. The degree verifies that

$$\text{deg}(a \circ b) = \text{deg } a + \text{deg } b. \tag{3.3}$$

This statement, according to definition (3.2), is obvious.

In order to enlighten the above construction of the \circ -product let us consider the following example. Let (M, ω) be a two-dimensional symplectic manifold. From the Darboux theorem (2.1) we can choose a chart (U_i, φ_i) in such a manner that at a fixed point $\mathbf{p} \in M$ and in some neighborhood of it we have that $\omega = dq^2 \wedge dq^1$. It means that a chart (U_i, φ_i) in a natural basis $\{\partial/\partial q^1, \partial/\partial q^2\}$ the tensor ω^{ij} takes the form

$$\omega^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let us consider the \circ -product of elements $a = a_{0,[11]}$ and $b = b_{0,[12]}$, where the symbol $[\dots]$ denotes the symmetrization in the indices inside the bracket. From (3.2) we obtain

$$\begin{aligned} a(X) \circ b(X) &= a_{0,11} X^1 X^1 \circ b_{0,12} X^1 X^2 \\ &= a_{0,11} b_{0,12} X^1 X^1 X^1 X^2 + \frac{1}{1!} \left(\frac{i\hbar}{2}\right) \cdot 1 \cdot 2a_{0,11} b_{0,12} X^1 X^1. \end{aligned}$$

So, finally

$$c_{0,[11,12]} = a_{0,11} b_{0,12}, \quad c_{1,[11]} = i \cdot a_{0,11} b_{0,12}.$$

We have worked until now at an arbitrary fixed point \mathbf{p} of the symplectic manifold (M, ω) . But our aim is to define the $*$ -product on the whole manifold. To achieve it we need to introduce a new object, so called the Weyl algebra bundle. For a definition of fibre bundle see, for instance, Ref. [85].

Definition 3.6. The collection of all the Weyl algebras, i.e.

$$\mathcal{P}^* \mathcal{M}[[\hbar]] = \bigcup_{\mathbf{p} \in M} \mathcal{P}^* \mathcal{M}_{\mathbf{p}}[[\hbar]]$$

is a vector bundle which is also an algebra bundle and it is called the Weyl algebra bundle.

Note that the structure of the Weyl algebra bundle looks as follows. The set $\mathcal{P}^*\mathcal{M}[[\hbar]]$ is the total space. The symplectic space (M, ω) is the base space. The fibre is the Weyl algebra $\mathcal{P}^*\mathcal{M}_{\mathbf{p}}[[\hbar]]$. We do not introduce a new symbol but now $\mathcal{P}^*\mathcal{M}_{\mathbf{p}}[[\hbar]]$ is not connected with any point. Moreover, $\mathcal{P}^*\mathcal{M}_{\mathbf{p}}[[\hbar]]$ is a Weyl vector space and $\omega^{ij}, i, j = 1, \dots, 2n$, are the coefficients of a fixed nondegenerate antisymmetric tensor. The projection map $\pi : \mathcal{P}^*\mathcal{M}[[\hbar]] \rightarrow M$ is defined by $\pi(v_{\mathbf{p}}[[\hbar]]) = \mathbf{p}$. The group $GL(2n, \mathbb{R})$ is the group of real automorphisms of $T_{\mathbf{p}}^*M$. The structure group of the fibre is

$$G := \bigoplus_{k=0}^{\infty} \bigoplus_{l=0}^{\infty} \underbrace{(GL(2n, \mathbb{R}) \otimes \dots \otimes GL(2n, \mathbb{R}))}_{l\text{-times}}.$$

For $l = 0$ we have just the identity transformation. The tensor ω^{ij} transforms under the group $GL(2n, \mathbb{R}) \otimes GL(2n, \mathbb{R})$. Moreover, if the tensor a_{ij} transforms under the element $g \in GL(2n, \mathbb{R}) \otimes GL(2n, \mathbb{R})$, then also ω^{ij} transforms under $g^{-1} \in GL(2n, \mathbb{R}) \otimes GL(2n, \mathbb{R})$. Let (U_i, ϕ_i) be a chart on M . The local trivialization ϕ_i is a diffeomorphism which assigns to every point of $\mathcal{P}^*\mathcal{M}[[\hbar]]$ the point \mathbf{p} of M and to the element of $\mathcal{P}^*\mathcal{M}_{\mathbf{p}}[[\hbar]]$ its coordinates in the natural basis $\{(\partial/\partial q_i) \otimes \dots \otimes (\partial/\partial q_i)\}$ determined by ϕ_i .

Let Λ^k be the vector bundle of k -forms on the symplectic manifold M . Taking the direct sum of tensor products of bundles

$$\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda = \bigoplus_{k=0}^{2n} \mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^k$$

we obtain a new algebra bundle. A product can be defined in it in terms of the \circ -product of elements of the Weyl algebra $\mathcal{P}^*\mathcal{M}_{\mathbf{p}}[[\hbar]]$ and the external product (\wedge) of forms. We will denote the new product also by ' \circ '. In a local chart (U_i, ϕ_i) the elements of $\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^k$ are smooth tensor fields of the kind

$$a_{c,i_1,\dots,i_l,j_1,\dots,j_k}(q^1, \dots, q^{2n}).$$

These objects $a_{c,i_1,\dots,i_l,j_1,\dots,j_k}$ are symmetric in the indices (i_1, \dots, i_l) (as elements of the Weyl algebra) and antisymmetric in (j_1, \dots, j_k) (as forms). For simplicity we will omit the coordinates (q^1, \dots, q^{2n}) . The elements of $\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda$ can be seen as forms with values in the Weyl algebra.

In the special case when $a \in \Lambda^0$ is a smooth function on M , we obtain

$$a \circ b = a \cdot b = a \wedge b \quad \forall b \in \mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda.$$

Definition 3.7 (PrzanowskiTosiek [74]). The commutator of two forms $a \in \mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{k_1}$ and $b \in \mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{k_2}$ is a form belonging to $\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{k_1+k_2}$ such that

$$[a, b] := a \circ b - (-1)^{k_1 \cdot k_2} b \circ a. \tag{3.4}$$

We point out some properties of the commutator of forms with values in the Weyl algebra:

1. The straightforward consequence of (3.4) is the equality

$$[b, a] = (-1)^{k_1 \cdot k_2 + 1} [a, b]. \tag{3.5}$$

2. The Jacobi identity

$$\begin{aligned} &(-1)^{k_2(k_1+k_3)} [a, [b, c]] + (-1)^{k_1(k_2+k_3)} [c, [a, b]] \\ &+ (-1)^{k_3(k_1+k_2)} [b, [c, a]] = 0 \end{aligned} \tag{3.6}$$

holds for every $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{k_1}$, $b \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{k_2}$ and $c \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{k_3}$. It is a simple matter of computation to prove this identity.

3. $[a, b \circ c] = (-1)^{k_1 k_2} b \circ [a, c] + [a, b] \circ c$ (3.7)

for all $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{k_1}$, $b \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{k_2}$ and $c \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{k_3}$.

Proof. Indeed,

$$[a, b \circ c] = a \circ b \circ c - (-1)^{k_1(k_2+k_3)} b \circ c \circ a. \tag{3.8}$$

Developing the r.h.s. of (3.7) we get

$$(-1)^{k_1 k_2} b \circ [a, c] = (-1)^{k_1 k_2} b \circ a \circ c - (-1)^{k_1 k_2} (-1)^{k_1 k_3} b \circ c \circ a \tag{3.9}$$

and

$$[a, b] \circ c = a \circ b \circ c - (-1)^{k_1 k_2} b \circ a \circ c. \tag{3.10}$$

Summing (3.9) and (3.10) we recover (3.8). \square

4. The commutator $[a, b]$ of two forms $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^m$ and $b \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^n$ contains only terms with an odd number of derivatives in X^i .
5. The commutator $[a, b]$ of two real forms $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^m$ and $b \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^n$ is purely imaginary.

Proofs of these last two properties can be found in [86].

4. Connections in the bundle $\mathcal{P}^* \mathcal{M}[[\hbar]] A$

Let (M, ω, ∇) be a Fedosov manifold equipped with a symplectic connection ∇ , whose coefficients in a Darboux atlas $\{(U_\rho, \varphi_\rho)\}_{\rho \in I}$ are Γ_{ijk} . The connection allows us to transport parallelly geometrical quantities on M .

Let us define the covariant derivative of a tensor $a_{t,i_1,\dots,i_l,j_1,\dots,j_k}$ with respect to the variable q^s . Following [85] we can write

$$\begin{aligned} \partial_{:s} a_{t,i_1,\dots,i_l,j_1,\dots,j_k} &:= \frac{\partial a_{t,i_1,\dots,i_l,j_1,\dots,j_k}}{\partial q^s} - \omega^{ur} \Gamma_{rsi_1} a_{t,u i_2,\dots,i_l,j_1,\dots,j_k} - \dots \\ &- \omega^{ur} \Gamma_{rsi_l} a_{t,i_1,\dots,i_{l-1}u,j_1,\dots,j_k} - \omega^{ur} \Gamma_{rsj_1} a_{t,i_1,\dots,i_l,u j_2,\dots,j_k} \\ &- \dots - \omega^{ur} \Gamma_{rsj_k} a_{t,i_1,\dots,i_l,j_1,\dots,j_{k-1}u}. \end{aligned}$$

The covariant derivative $\partial_{;s}$ increases the covariance of the tensor $a_{t,i_1,\dots,i_l,j_1,\dots,j_k}$, so the object $\partial_{;s}a_{t,i_1,\dots,i_l,j_1,\dots,j_k}$ is a tensor of the range $(0, l + k + 1)$. On the other hand, the covariant derivative $\partial_{;s}a_{t,i_1,\dots,i_l,j_1,\dots,j_k}$ is again symmetric in indices (i_1, \dots, i_l) and antisymmetric in (j_1, \dots, j_k) . The index ‘s’ does not generate new symmetries in $\partial_{;s}a_{t,i_1,\dots,i_l,j_1,\dots,j_k}$. The covariant derivative is nonsymmetric in (s, i_1, \dots, i_l) and nonantisymmetric in (s, j_1, \dots, j_k) . The straightforward consequence of this fact is that $\partial_{;s}a_{t,i_1,\dots,i_l,j_1,\dots,j_k}$ is not an element of $\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda$.

Such a result is not compatible with our expectations. But there exists a linear operator known as the exterior covariant derivative operator [87], which transforms k -forms with values in the Weyl algebra in $(k + 1)$ -forms with values in $\mathcal{P}^*\mathcal{M}[[\hbar]]$.

Definition 4.1. The exterior covariant derivative $\partial : \mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^k \rightarrow \mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{k+1}$ is a linear differential operator such that for every $a \in \mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda$

$$\partial a := dq^s \wedge \partial_{;s}a.$$

Note that in Fedosov’s publications [67,68] the operator ∂ is simply called ‘connection’.

Theorem 4.1 (Fedosov [68]). *The exterior covariant derivative ∂ in the bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda$ in a Darboux atlas $\{(U_\varrho, \varphi_\varrho)\}_{\varrho \in I}$ can be written as*

$$\partial a = da + \frac{1}{i\hbar}[G, a], \tag{4.1}$$

where $a \in \mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda$ and

$$G := \frac{1}{2}\Gamma_{i,j,k}dq^k \tag{4.2}$$

is a 1-form symmetric in (i, j, k) .

It is worth making the following remarks:

1. The formula (4.1) is valid only in Darboux coordinates, because only in such charts Γ_{ijk} is symmetric in all the indices.
2. We consider G like an element of $\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^1$ although we know that the elements Γ_{ijk} are not components of any tensor. This fact does not influence on the \circ -product in a Darboux chart.
3. The formula (4.1) is similar to the definition of the connection matrix [88] but, in fact, the 1-form (4.2) is not the connection matrix. In the analyzed case the connection matrix is infinite dimensional.

Theorem 4.2 (Fedosov [68]). *The exterior covariant derivative preserves the degree of the forms of $\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda$.*

Proof. The derivative d does not change the degree of the forms of $\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda$. The commutator increases the power of \hbar in one unit since it has two derivatives in X^i ’s, but the fact of dividing by $i\hbar$ finally preserves the degree.

Theorem 4.3. For every two forms $a \in \mathcal{P}^* \mathcal{M}[\hbar] \otimes \Lambda^{k_1}$ and $b \in \mathcal{P}^* \mathcal{M}[\hbar] \otimes \Lambda^{k_2}$ we have

$$\partial(a \circ b) = \partial a \circ b + (-1)^{k_1} a \circ \partial b. \tag{4.3}$$

Proof. We make the proof in Darboux coordinates.

$$\begin{aligned} \partial(a \circ b) &:= d(a \circ b) + \frac{1}{i\hbar} [\Gamma, a \circ b] \stackrel{(3.8)}{=} da \circ b + (-1)^{k_1} a \circ db \\ &+ \frac{1}{i\hbar} (-1)^{1 \cdot k_1} a \circ [\Gamma, b] + \frac{1}{i\hbar} [\Gamma, a] \circ b = \partial a \circ b + (-1)^{k_1} a \circ \partial b. \quad \square \end{aligned}$$

Note that the relation (4.3) is a Leibniz rule for the exterior covariant derivative ∂ and the \circ -product.

Let us consider now the second exterior covariant derivative $\partial(\partial a)$. As before we make our computations in a Darboux chart. Thus,

$$\partial(\partial a) = \partial \left(da + \frac{1}{i\hbar} [\Gamma, a] \right) = d(da) + d \left(\frac{1}{i\hbar} [\Gamma, a] \right) + \frac{1}{i\hbar} [\Gamma, da + \frac{1}{i\hbar} [\Gamma, a]]. \tag{4.4}$$

Using the following general property valid for any form

$$d(a \wedge b) = da \wedge b + (-1)^k a \wedge db, \quad a \in \Lambda^k$$

and remembering that Γ is a 1-form, we get

$$d(\Gamma \circ a) = d\Gamma \circ a - \Gamma \circ da, \quad d(a \circ \Gamma) = da \circ \Gamma + (-1)^k a \circ d\Gamma. \tag{4.5}$$

From (3.4) and (4.5) we obtain

$$d[\Gamma, a] = [d\Gamma, a] - [\Gamma, da]. \tag{4.6}$$

From the Jacobi identity (3.6)

$$(-1)^{1 \cdot (1+k)} [\Gamma, [\Gamma, a]] + (-1)^{1 \cdot (1+k)} [a, [\Gamma, \Gamma]] + (-1)^{k \cdot (1+1)} [\Gamma, [a, \Gamma]] = 0$$

and using the relation

$$[[\Gamma, [a, \Gamma]] \stackrel{(3.5)}{=} (-1)^{1 \cdot (1+k)} [\Gamma, [\Gamma, a]],$$

we see that

$$2[\Gamma, [\Gamma, a]] = -[a, [\Gamma, \Gamma]] \stackrel{(3.5)}{=} [[\Gamma, \Gamma], a]. \tag{4.7}$$

Moreover,

$$[\Gamma, \Gamma] = 2\Gamma \circ \Gamma. \tag{4.8}$$

Putting (4.6) and (4.7) with (4.8) in (4.4) we get

$$\partial(\partial a) = \frac{1}{i\hbar} [d\Gamma + \frac{1}{i\hbar} \Gamma \circ \Gamma, a].$$

We see that the 2-form

$$R = \Gamma + \frac{1}{i\hbar} \Gamma \circ \Gamma \tag{4.9}$$

is the curvature of the connection Γ . In Darboux charts we can write

$$\partial(\partial a) = \frac{1}{i\hbar} [R, a]. \tag{4.10}$$

Let us find the explicit form of the 2-form of curvature (4.9). We consider elements of the Weyl algebra bundle $\mathcal{P}^* \mathcal{M}[[\hbar]]$ acting on vectors. By X we denote an arbitrary fixed vector field on M . Hence

$$\begin{aligned} d\Gamma(X, X) &= d \left(\frac{1}{2} \Gamma_{ijk} X^i X^j dq^k \right) \\ &= \frac{1}{2} \frac{\partial \Gamma_{ijk}}{\partial q^l} X^i X^j dq^l \wedge dq^k = \frac{1}{4} \left(\frac{\partial \Gamma_{ijk}}{\partial q^l} - \frac{\partial \Gamma_{ijl}}{\partial q^k} \right) X^i X^j dq^l \wedge dq^k \\ &= \frac{1}{4} \omega_{is} \left(\frac{\partial \Gamma_{kj}^s}{\partial q^l} - \frac{\partial \Gamma_{lj}^s}{\partial q^k} \right) X^i X^j dq^l \wedge dq^k \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \Gamma \circ \Gamma(X, X) &:= \frac{i\hbar}{2} \omega^{i_1 j_1} \cdot 2 \cdot 2 \cdot \frac{1}{2} \Gamma_{i_1 i l} X^i dq^l \wedge \frac{1}{2} \Gamma_{j_1 j k} X^j dq^k \\ &= \frac{i\hbar}{2} \omega^{i_1 j_1} \omega_{i_1 u} \Gamma_{i l}^u \Gamma_{j_1 j k} X^i X^j dq^l \wedge dq^k \\ &= -\frac{i\hbar}{2} \Gamma_{i l}^{j_1} \Gamma_{j_1 j k} X^i X^j dq^l \wedge dq^k = -\frac{i\hbar}{2} \omega_{js} \Gamma_{i l}^u \Gamma_{u k}^s X^i X^j dq^l \wedge dq^k \\ &= \frac{i\hbar}{4} \omega_{is} (\Gamma_{jk}^u \Gamma_{ul}^s - \Gamma_{jl}^u \Gamma_{uk}^s) X^i X^j dq^l \wedge dq^k. \end{aligned} \tag{4.12}$$

From (4.11) and the last expression of (4.12) we obtain that

$$R(X, X) = \frac{1}{4} R_{ijkl} X^i X^j dq^l \wedge dq^k, \tag{4.13}$$

where

$$R_{ijkl} = \omega_{si} \left(\frac{\partial \Gamma_{kj}^s}{\partial q^l} - \frac{\partial \Gamma_{lj}^s}{\partial q^k} + \Gamma_{jk}^u \Gamma_{ul}^s - \Gamma_{uk}^s \Gamma_{jl}^u \right)$$

is the curvature tensor (2.8) of the symplectic connection Γ_{ijk} on M . From (4.13) we see that R_{ijkl} is symmetric in the indices $\{i, j\}$ and antisymmetric in $\{k, l\}$.

It can be proved that although we work in a Darboux chart, the relations (4.13) and (4.10) hold in every chart on \mathcal{M} .

Let us introduce two operators acting on the bundle $\mathcal{P}^* \mathcal{M}[[\hbar]] \Lambda$. To make the notation more clear we will operate on elements of the Weyl algebra acting on vectors of the tangent space TM to the symplectic manifold M , i.e. $a_{i_1, \dots, i_l, j_1, \dots, j_s} dq^{j_1} \wedge \dots \wedge dq^{j_s} (X^1, \dots, X^l)$.

Definition 4.2. The operator $\delta : \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^s \rightarrow \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{s+1}$ defined as

$$\delta a := dq^k \wedge \left(\frac{\partial a}{\partial X^k} \right)$$

is known as antiderivation.

Theorem 4.4. The operator δ may be written as a 1-form

$$\delta = -\frac{1}{i\hbar} [\omega_{ij} X^i dq^j, \cdot]. \tag{4.14}$$

Theorem 4.5. The operator δ lowers the degree of the elements of $\mathcal{P}^* \mathcal{M}[[\hbar]] \Lambda$ of 1.

Proofs of both theorems are straightforward.

Definition 4.3. The operator $\delta^* : \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^s \rightarrow \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{s-1}$ is defined as

$$\delta^* a := X^k \left(\frac{\partial}{\partial q^k} \right) \lrcorner a.$$

It can be considered as the ‘opposite’ of the antiderivation operator. The symbol \lrcorner denotes the inner product.

Theorem 4.6 (Fedosov [68]). The operator δ^* raises the degree of the forms of $\mathcal{P}^* \mathcal{M}[[\hbar]] \Lambda$ in 1.

Proof. The operator δ^* exchanges the 1-form dq^k into X^k and does not touch \hbar , so the degree increases in 1. \square

Theorem 4.7 (Fedosov [68]). The operators δ and δ^* do not depend on the choice of local coordinates and have the following properties:

- (i) linearity;
- (ii) $\delta^2 = (\delta^*)^2 = 0$;
- (iii) let us assume that indices i_1, \dots, j_s are arbitrary but fixed. For the monomial $X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_s}$ we have

$$(\delta\delta^* + \delta^*\delta)X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_s} = (l + s)X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_s}.$$

Proof. Linearity is obvious, thus let us start from the second property. From the symmetry of $X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_s}$ in X^i 's it is sufficient to compute δ^2 for two fixed X 's. Hence

$$\delta^2(X^{i_1} X^{i_2}) = \delta(X^{i_2} dq^{i_1} + X^{i_1} dq^{i_2}) = dq^{i_2} \wedge dq^{i_1} + dq^{i_1} \wedge dq^{i_2} = 0.$$

Now from the antisymmetry of $X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_s}$ in dq^j 's it is enough to find $(\delta^*)^2(dq^{j_1} \wedge dq^{j_2})$ for two fixed dq^{j_1} and dq^{j_2} . So,

$$(\delta^*)^2(dq^{j_1} \wedge dq^{j_2}) = \delta^*(X^{j_1} dq^{j_2} - X^{j_2} dq^{j_1}) = X^{j_1} X^{j_2} - X^{j_2} X^{j_1} = 0.$$

To show the third statement we consider the monomial $X^i dq^j$. The result is easy to generalize on $X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_s}$ using the symmetry in X^i 's and the antisymmetry in dq^j 's. Now

$$\begin{aligned} (\delta\delta^* + \delta^*\delta)(X^i dq^j) &= \delta(X^i X^j) + \delta^*(dq^i \wedge dq^j) \\ &= X^i dq^j + X^j dq^i + X^i dq^j - X^j dq^i = 2X^i dq^j. \quad \square \end{aligned}$$

Definition 4.4 (Fedosov [68]). There is an operator $\delta^{-1} : \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^s \rightarrow \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{s-1}$ defined by

$$\delta^{-1} a := \begin{cases} \frac{1}{l+s} \delta^* a & \text{for } l+s > 0, \\ 0 & \text{for } l+s = 0, \end{cases} \tag{4.15}$$

where l is the degree of a in X^i 's (i.e. the number of X 's) and s is the degree of the form.

Note that, in fact, δ^{-1} is, up to a constant, the δ^* operator.

The straightforward consequence of the linearity and the decomposition on monomials is the de Rham decomposition of the form $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \Lambda$ as shows next theorem.

Theorem 4.8 (Fedosov [68]). For every $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \Lambda$ the equality holds

$$a = \delta\delta^{-1} a + \delta^{-1} \delta a + a_{00}, \tag{4.16}$$

where a_{00} is a function on the symplectic manifold M .

Proof. For functions a_{00} on M the fact that the relation (4.16) holds is evident. To show that the de Rham decomposition is true also for elements of $\mathcal{P}^* \mathcal{M}[[\hbar]] \Lambda$ we use the fact that δ and δ^{-1} are linear operators. From Theorem 4.7

$$(\delta\delta^* + \delta^*\delta)(X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_s}) = (l+s) X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_s}. \tag{4.17}$$

From definition (4.15) the l.h.s. of the above equation may be written as

$$(\delta(l+s)\delta^{-1} + ((l+1) + (s-1))\delta^{-1}\delta) X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_s}.$$

Hence, from (4.17) we immediately obtain (4.16). \square

The definition (4.1) of the exterior covariant derivative is based on covariant derivatives determined by the symplectic connection. It is possible to generalize this description of exterior covariant derivative (compare with [89] where the term ‘connection’ is used instead of ‘exterior covariant derivative’).

Definition 4.5. The exterior covariant derivative $\check{\delta} : \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^0 \rightarrow \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^1$ is a linear differential operator such that for every $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^0$ and $f \in \Lambda^0$

$$\check{\delta}(f \cdot a) = df \otimes a + f \cdot \check{\delta}a. \tag{4.18}$$

This definition may be extended to an arbitrary k -form with values in $\mathcal{P}^* \mathcal{M}[[\hbar]]$.

Theorem 4.9 ([89]). *There is a unique operator $\check{\delta} : \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^k \rightarrow \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{k+1}$ satisfying:*

$$1. \quad \check{\delta}(f \wedge a) = df \wedge a + (-1)^s f \wedge \check{\delta}a \tag{4.19}$$

for every $f \in \Lambda^s, a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^k$,

$$2. \quad \check{\delta}a = \tilde{\delta}a \tag{4.20}$$

for $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^0$.

In a local Darboux chart we can write

$$\check{\delta}a = da + \frac{1}{i\hbar} [\Phi, a], \tag{4.21}$$

where $\Phi \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^1$ is a connection determining the derivation $\check{\delta}$. The same definition works for 0-forms, so if $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^0$ then

$$\tilde{\delta}a = da + \frac{1}{i\hbar} [\Phi, a].$$

Indeed, the operator $d + \frac{1}{i\hbar} [\Phi, \cdot]$ is linear. For every $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^0$ and $f \in \Lambda^0$

$$\begin{aligned} \check{\delta}a &= d(f \cdot a) + \frac{1}{i\hbar} [\Phi, f \cdot a] = df \cdot a + (-1)^s f \wedge da + f \cdot \frac{1}{i\hbar} [\Phi, a] \\ &= df \cdot a + f \cdot \tilde{\delta}a. \end{aligned}$$

Hence, the condition (4.18) is fulfilled. Now for $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^k$ and $f \in \Lambda^s$ we see from (4.21) that the property (4.19) holds, i.e.

$$\begin{aligned} \check{\delta}(f \wedge a) &= d(f \wedge a) + \frac{1}{i\hbar} [\Phi, f \wedge a] \stackrel{(3.4)}{=} df \wedge a + (-1)^s f \wedge da \\ &\quad + \frac{1}{i\hbar} \Phi \circ (f \wedge a) - (-1)^{1+(s+k)} \frac{1}{i\hbar} (f \wedge a) \circ \Phi \\ &= df \wedge a + (-1)^s f \wedge da + (-1)^s f \wedge \frac{1}{i\hbar} \Phi \circ a - (-1)^s f \wedge (-1)^k \frac{1}{i\hbar} a \circ \Phi \\ &= df \wedge a + (-1)^s f \wedge \left(da + \frac{1}{i\hbar} [\Phi, a] \right) = df \wedge a + (-1)^s f \wedge \tilde{\delta}a. \end{aligned}$$

Since $\tilde{\delta}a = \check{\delta}a$ for every $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^0$ we can use the formula (4.21) as definition of the exterior covariant derivative in $\mathcal{P}^* \mathcal{M}[[\hbar]] \Lambda$.

An example of the connection Φ is the connection appearing in the Fedosov paper [90]

$$\Phi = \omega_{ij} X^i dq^j + a_{i_1 i_2, j} X^{i_1} X^{i_2} dq^j + \dots$$

Analogously to the case of the connection Γ (4.9) we define the curvature of Φ as a 2-form with values in the Weyl algebra

$$\check{R} = d\Phi + \frac{1}{i\hbar} \Phi \circ \Phi. \tag{4.22}$$

Of course, in Darboux coordinates for every $a \in \mathcal{P}^* \mathcal{M}[[\hbar]] \Lambda$ we can write

$$\check{\partial}(\check{\partial}a) = \frac{1}{i\hbar} [\check{R}, a]. \tag{4.23}$$

Among the properties of the curvature \check{R} we present the so-called Bianchi identity.

Theorem 4.10. *For the curvature \check{R} the following relation holds*

$$\check{\partial}\check{R} = 0. \tag{4.24}$$

Proof.

$$\begin{aligned} \check{\partial}\check{R} &\stackrel{(4.21)}{=} d\check{R} + \frac{1}{i\hbar} [\Phi, \check{R}] \stackrel{(4.22)}{=} d \left(d\Phi + \frac{1}{i\hbar} \Phi \circ \Phi \right) + \frac{1}{i\hbar} \left[\Phi, d\Phi + \frac{1}{i\hbar} \Phi \circ \Phi \right] \\ &= d^2\Phi + \frac{1}{i\hbar} d\Phi \circ \Phi + (-1)^1 \frac{1}{i\hbar} \Phi \circ d\Phi + \frac{1}{i\hbar} \Phi \circ d\Phi - \frac{1}{i\hbar} (-1)^2 d\Phi \circ \Phi \\ &\quad + \frac{1}{(i\hbar)^2} [\Phi, \Phi \circ \Phi] = \frac{1}{(i\hbar)^2} \left(\Phi \circ \Phi \circ \Phi - (-1)^{1 \cdot 2} \Phi \circ \Phi \circ \Phi \right) = 0. \quad \square \end{aligned}$$

In our considerations a special role will be played by the so called ‘abelian connection’.

Definition 4.6 (Fedosov [68]). A connection D (defined by (4.20)) in the bundle $\mathcal{P}^* \mathcal{M}[[\hbar]] \Lambda$ is called abelian if for any section $a \in C^\infty(\mathcal{P}^* \mathcal{M}[[\hbar]] \Lambda)$

$$D^2 a = \frac{1}{i\hbar} [\Omega, a] = 0.$$

The curvature Ω of the abelian connection D is a central form. Since Ω is central we deduce that

$$d\Omega = 0.$$

Indeed, from the general formula (4.21) we have

$$D\Omega = d\Omega + \frac{1}{i\hbar} [\check{R}, \Omega],$$

where by $\tilde{\Gamma}$ we denote the 1-form of the abelian connection. Since Ω is a central form the commutator $[\tilde{\Gamma}, \Omega]$ disappears. From the other side, the Bianchi identity (4.24) holds for the curvature Ω , i.e. $D\Omega = 0$. Hence $d\Omega = 0$.

Let us assume that in a Darboux chart

$$D = \partial + \frac{1}{i\hbar}[(\omega_{ij}X^i dq^j + r), \cdot], \tag{4.25}$$

where r is a globally defined 1-form satisfying the Weyl normalizing condition, i.e. the part of r not containing X^i 's vanishes. Let us compute the curvature of the connection (4.25). As we have mentioned before in (4.22)

$$\Omega = d\tilde{\Gamma} + \frac{1}{i\hbar}\tilde{\Gamma} \circ \tilde{\Gamma},$$

where

$$\tilde{\Gamma} := \Gamma + \omega_{ij}X^i dq^j + r. \tag{4.26}$$

We can see that

$$\begin{aligned} \Omega &= d\Gamma + dr + \frac{1}{i\hbar}\Gamma \circ \Gamma + \frac{1}{i\hbar}\Gamma \circ \omega_{ij}X^i dq^j + \frac{1}{i\hbar}\Gamma \circ r + \frac{1}{i\hbar}\omega_{ij}X^i dq^j \circ \Gamma \\ &\quad + \frac{1}{i\hbar}\omega_{ij}X^i dq^j \circ \omega_{kl}X^k dq^l + \frac{1}{i\hbar}\omega_{ij}X^i dq^j \circ r + \frac{1}{i\hbar}r \circ \Gamma + \frac{1}{i\hbar}r \circ \omega_{ij}X^i dq^j \\ &\quad + \frac{1}{i\hbar}r \circ r. \end{aligned} \tag{4.27}$$

We know that

$$d\Gamma + \frac{1}{i\hbar}\Gamma \circ \Gamma = R. \tag{4.28}$$

Since our definition has been formulated in a Darboux chart we can write

$$dr + \frac{1}{i\hbar}\Gamma \circ r + \frac{1}{i\hbar}r \circ \Gamma = dr + \frac{1}{i\hbar}[r, \Gamma] = \partial r. \tag{4.29}$$

Moreover,

$$\begin{aligned} \frac{1}{i\hbar}\Gamma \circ \omega_{ij}X^i dq^j + \frac{1}{i\hbar}\omega_{ij}X^i dq^j \circ \Gamma &= \frac{1}{i\hbar}[\Gamma, \omega_{ij}X^i dq^j] \\ &= \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{1}{2}\omega^{li}\Gamma_{lrs}X^r dq^s \wedge \omega_{ij}dq^j. \end{aligned} \tag{4.30}$$

We know that (2.3) holds, hence (4.30) equals to

$$\delta_j^l \Gamma_{lrs}X^r dq^s \wedge dq^j = \Gamma_{lrs}X^r dq^s \wedge dq^l = 0, \tag{4.31}$$

because Γ_{lrs} is symmetric in all the indices. Now

$$\frac{1}{i\hbar}\omega_{ij}X^i dq^j \circ r + \frac{1}{i\hbar}r \circ \omega_{ij}X^i dq^j = \frac{1}{i\hbar}[\omega_{ij}X^i dq^j, r] \stackrel{(4.14)}{=} -\delta r. \tag{4.32}$$

Finally

$$\frac{1}{i\hbar}\omega_{ij}X^i dq^j \circ \omega_{kl}X^k dq^l = -\frac{1}{2}\omega_{ij}dq^i \wedge dq^j. \tag{4.33}$$

Putting (4.28)–(4.33) into (4.27) we obtain

$$\Omega = -\frac{1}{2}\omega_{ij}dq^i \wedge dq^j + R - \delta r + \partial r + \frac{1}{i\hbar}r \circ r. \tag{4.34}$$

The abelian property will be fulfilled, provided

$$\delta r = R + \partial r + \frac{1}{i\hbar}r \circ r. \tag{4.35}$$

Theorem 4.11 (Fedosov [68]). *The equation (4.35) has a unique solution satisfying*

$$\text{deg } r \geq 3, \quad \delta^{-1}r = 0.$$

Proof. From the decomposition (4.16) we obtain

$$r = \delta^{-1}\delta r,$$

because r is a 1-form we have $r_{00} = 0$ and $\delta^{-1}r = 0$. Moreover, δr is a solution of (4.35). Hence

$$r = \delta^{-1}(R + \partial r + \frac{1}{i\hbar}r \circ r). \tag{4.36}$$

The operator δ^{-1} raises the degree by 1, so (4.36) is a recurrent formula starting with $\delta^{-1}R$. The proof that solution (4.36) really fulfills (4.35) and that it is unique, is more complicated (see it in [68] or [74]). \square

Note that we have not included in the expression of r terms with degrees lower than 3. This is due to the fact that the abelian connection \tilde{F}^1 (4.26) contains terms with degrees 1 and 2 and for that r , from its definition, is an object with degree $\text{deg } r \geq 3$.

Theorem 4.12. *For the abelian connection D and two forms $a \in \mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{k_1}$ and $b \in \mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{k_2}$ we have*

$$D(a \circ b) = Da \circ b + (-1)^{k_1}a \circ Db.$$

The proof is analogous to that of Theorem 4.3.

Corollary 4.1 (Fedosov [68]). *The set of 0-forms such that their abelian connection vanishes forms the subalgebra $\mathcal{P}^*\mathcal{M}_D[[\hbar]]$ of $\mathcal{P}^*\mathcal{M}[[\hbar]]$.*

5. The $*$ -product on M

In former sections we have studied the structure of the Weyl algebra bundle. Thus, we are able to introduce the $*$ -product on the symplectic manifold M .

Definition 5.1. The projection $\sigma : \mathcal{P}^* \mathcal{M}[[\hbar]] \rightarrow C^\infty(M)$ assigns to each 0-form a of $\mathcal{P}^* \mathcal{M}[[\hbar]]$ its part a_{00} (according to its de Rham decomposition (4.16)), i.e. $\sigma(a) := a_{00}$.

The following theorem holds.

Theorem 5.1 (Fedosov [68]). *For any function $a_{00} \in C^\infty(M)$ there exists a unique section a of $C^\infty(\mathcal{P}^* \mathcal{M}_D[[\hbar]])$ such that $\sigma(a) = a_{00}$. The element a is defined by the recurrent formula*

$$a = a_{00} + \delta^{-1} \left(\partial a + \frac{1}{i\hbar} [r, a] \right).$$

The recurrent character of that solution can be proved analogously to (4.36) (for more details see [68]).

The map σ gives a one-to-one correspondence between $C^\infty(\mathcal{P}^* \mathcal{M}_D[[\hbar]])$ and $C^\infty(M)$. Then $a = \sigma^{-1}(a_{00})$, where σ^{-1} is the inverse map of σ . This map provides a quantization procedure as follows.

Definition 5.2 (Fedosov [68]). Let F_1, F_2 be two $C^\infty(M)$ -functions. The $*$ -product is defined as

$$F_1 * F_2 := \sigma(\sigma^{-1}(F_1) \circ \sigma^{-1}(F_2)). \tag{5.1}$$

This $*$ -product can be considered as a generalization of the Moyal product of Weyl type defined for $M = \mathbb{R}^{2n}$. It has the following properties:

1. Invariance under Darboux transformations. In fact, the $*$ -product is invariant under all smooth transformation of coordinates on M . However, the definitions of exterior covariant derivatives ∂ and D cannot be written in the form (4.1) and (4.25), respectively.
2. In the limit $\hbar \rightarrow 0^+$ the $*$ -product turns into the commutative point-wise multiplication of functions, i.e.

$$\lim_{\hbar \rightarrow 0^+} F_1 * F_2 = F_1 \cdot F_2. \tag{5.2}$$

Indeed, since in (5.1) only terms not containing X^i 's and $\hbar^n, n \geq 1$, are taking into account we can see that, in fact, only the terms of degree zero are essential in the product $\sigma^{-1}(F_1) \circ \sigma^{-1}(F_2)$. From the third property of the \circ -product (3.3) we deduce that the part of $\sigma^{-1}(F_1) \circ \sigma^{-1}(F_2)$ with degree zero is just $F_1 \cdot F_2$. Hence, we obtain (5.2).

3. The multiplication (5.1) is associative but noncommutative.

Effectively, from theorem (5.1) the relation

$$\sigma^{-1}(F_1) = \sigma^{-1}(F_2) \iff F_1 = F_2$$

holds. It means that

$$\sigma^{-1}((F_1 * F_2) * F_3) = \sigma^{-1}(F_1 * F_2) \circ \sigma^{-1}(F_3) = (\sigma^{-1}(F_1) \circ \sigma^{-1}(F_2)) \circ \sigma^{-1}(F_3). \tag{5.3}$$

Since the \circ -product is associative the r.h.s. of (5.3) can be written as

$$\begin{aligned} \sigma^{-1}(F_1) \circ (\sigma^{-1}F_2) \circ \sigma^{-1}(F_3) &= \sigma^{-1}(F_1) \circ \sigma^{-1}((F_2 * F_3)) \\ &= \sigma^{-1}(F_1 * (F_2 * F_3)). \end{aligned}$$

Hence

$$(F_1 * F_2) * F_3 = F_1 * (F_2 * F_3) \quad \forall F_1, F_2, F_3 \in C^\infty(M).$$

4. When $M = \mathbb{R}^{2n}$ the product defined above is just the Moyal product of Weyl type. Instead of proving this statement we will analyze this case in the next section.

The properties mentioned above show that the $*$ -product constructed according to the Fedosov idea is a natural generalization of the $*$ -product in the trivial case when $M = \mathbb{R}^{2n}$ (Moyal-product). Several different properties of the $*$ -product may be found in [70] or [86].

In this paper we have only considered $*$ -products of the Weyl type. Other kinds of $*$ -products, whose geometrical origin is the same that of the $*$ -product of the Weyl type, were analyzed in [74].

The existence of many $*$ -products on the same symplectic manifold is closely related to the equivalence problem of $*$ -products. Two $*$ -products $*_1$ and $*_2$ are said to be equivalent iff there exists a differential operator \hat{T} such that for every two functions F_1, F_2 of $C^\infty(M)$, for which expressions appearing below have sense, the following relation holds

$$F_1 *_1 F_2 = \hat{T}^{-1}(\hat{T}F_1 *_2 \hat{T}F_2). \tag{5.4}$$

It has been proved in [91] that all the $*$ -products on a symplectic manifold are equivalent to the Weyl type $*$ -product constructed according to the Fedosov recipe. For more details about the equivalence problem of $*$ -products see [91–93].

The definition of the $*$ -product is not sufficient to study completely most of the physical problems. To solve the eigenvalue equation for an observable or to find its average value we must know how to define the states. There is not a general answer to this question. It seems that in the framework of the Fedosov formalism the states are described by functionals W over some functions defined on the phase space of the system but, in fact, knowledge about such objects is rather poor.

Topics connected with the representation problem of a quantum state on a phase space such as ‘traciality’ or closeness of the $*$ -product can be found, for instance, in [68,94,95].

The Fedosov algorithm is not the only way to define $*$ -product in nontrivial spaces. Several examples of $*$ -products on nontrivial spaces may be found in [96].

6. Examples

In this section we present three systems where the Fedosov construction is accomplished. The results obtained in two of them are known from the ‘traditional’ quantum mechanics. The third one, recently developed by us [86], does not have its counterpart in the formalism of operators in a certain Hilbert space.

6.1. Cotangent bundle $T^*\mathbb{R}^n$

In this first example, we consider the simple case of a physical system whose phase space is the vector space $(\mathbb{R}^{2n}, \omega)$. This space is covered with one chart $(\mathbb{R}^{2n}, \varrho)$, in which the symplectic form takes its natural shape, i.e.

$$\omega = \sum_{i=1}^n dq^{n+i} \wedge dq^i. \tag{6.1}$$

We choose the symplectic connection in such a way that all the coefficients vanish on \mathbb{R}^{2n} , i.e. $\Gamma_{ijk} = 0, 1 \leq i, j, k \leq 2n$. Hence the symplectic curvature tensor R_{ijkl} disappears.

The abelian connection (4.26) in the Weyl bundle is given by

$$\tilde{\Gamma} = \omega_{ij} X^i dq^j$$

and its curvature is a central form (4.34)

$$\Omega = -\frac{1}{2} \sum_{i=1}^n dq^{n+i} \wedge dq^i.$$

Hence, $r = 0$. It means that for every $F \in C^\infty(\mathbb{R}^{2n})$ according to Theorem 5.1 we can write

$$\sigma^{-1}(F) = F + \delta^{-1}(dF) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m F}{\partial q^{i_1} \dots \partial q^{i_m}} X^{i_1} \dots X^{i_m}.$$

From (1.3), (3.2) and (6.1) we can see that

$$\sigma^{-1}(F_1) \circ \sigma^{-1}(F_2) = F_1(Q + X) *_W F_2(Q + X), \tag{6.2}$$

where $Q = (q^1, \dots, q^{2n})$ and $X = (X^1, \dots, X^{2n})$. The symbol ‘ $*_W$ ’ denotes the $*$ -product (1.3) of Weyl type on \mathbb{R}^{2n} . We can write the equality relation in (6.2) because \mathbb{R}^{2n} and $T\mathbb{R}^{2n}$ are isomorphic.

Now from (5.1) we obtain that

$$F_1 * F_2 = \sigma(F_1(Q + X) *_W F_2(Q + X)) = F_1(Q) *_W F_2(Q).$$

We conclude that in case the symplectic space is just $(\mathbb{R}^{2n}, \omega)$, the $*$ -product computed using the Fedosov method is exactly the usual $*$ -product of the Weyl type.

6.2. Harmonic oscillator

Let us consider a system with phase space $(\mathbb{R}^{2n}, \omega)$, where $\omega = dp \wedge dq$, and Hamiltonian

$$H = \frac{p^2}{2m} + \frac{mq^2}{2}.$$

In a chart with coordinates (p, q) all the coefficients of the connection $\Gamma_{ijk}, i, j, k = 1, 2$, vanish. In the new Darboux coordinates (H, ϕ) , related with the old ones (p, q) according to

$$q = \sqrt{\frac{2H}{m}} \sin \phi, \quad p = \sqrt{2mH} \cos \phi,$$

the symplectic form is rewritten as

$$\omega = dH \wedge d\phi.$$

From the transformation rule for connections

$$\Gamma_{ijk}(Q) = \Gamma_{rst}(q) \frac{\partial q^r}{\partial Q^i} \frac{\partial q^s}{\partial Q^j} \frac{\partial q^t}{\partial Q^k} + \omega_{rs}(q) \frac{\partial q^s}{\partial Q^i} \frac{\partial^2 q^r}{\partial Q^j \partial Q^k},$$

we immediately obtain that the symplectic connection 1-form in coordinates (H, ϕ) is given by

$$\Gamma = \frac{1}{4H} X^2 X^2 d\phi + \frac{1}{2H} X^1 X^2 dH + H X^1 X^1 dH.$$

Obviously, its curvature vanishes. That implies that also $r = 0$ (see (4.36)) and the symplectic curvature in the Weyl algebra bundle is

$$\Omega = -\frac{1}{2} dH \wedge d\phi.$$

Moreover, the abelian connection is

$$\tilde{\Gamma} = X^1 d\phi - X^2 dH + \Gamma.$$

Let us look for the eigenvalues and eigenfunctions of the Hamiltonian H in coordinates (H, ϕ) . The eigenvalue equation takes the form

$$H * W_E(H, \phi) = E \cdot W_E(H, \phi),$$

where $W_E(H, \phi)$ is a functional representation of an eigenstate with eigenvalue E . From Theorem 5.1 we can write

$$\sigma^{-1}(H) = H + X^2 + HX^1X^1 + \frac{1}{4H}X^2X^2$$

and

$$\begin{aligned} \sigma^{-1}(W_E) &= W_E + \frac{\partial W_E}{\partial \phi} X^1 + \frac{\partial W_E}{\partial H} X^2 + H \frac{\partial W_E}{\partial H} X^1 X^1 + \frac{1}{2} \frac{\partial^2 W_E}{\partial \phi^2} X^1 X^1 \\ &\quad - \frac{1}{2H} \frac{\partial W_E}{\partial \phi} X^1 X^2 + \frac{\partial^2 W_E}{\partial \phi \partial H} X^1 X^2 + \frac{1}{4H} \frac{\partial W_E}{\partial H} X^2 X^2 + \frac{1}{2} \frac{\partial^2 W_E}{\partial H^2} X^2 X^2 \\ &\quad + \frac{1}{6} \frac{\partial^3 W_E}{\partial \phi^3} X^1 X^1 X^1 + \dots \end{aligned}$$

The series $\sigma^{-1}(W_E)$ is infinite but only the terms of $\text{deg} \leq 2$ are essential because $\sigma^{-1}(H)$ has degree 2. Computing $\sigma^{-1}(H) \circ \sigma^{-1}(W_E)$ and projecting the product on the phase space we finally obtain that

$$\begin{aligned} H * W_E(H, \phi) &= (H - E)W_E - \frac{\hbar^2}{4} \frac{\partial W_E}{\partial H} - \frac{\hbar^2}{4} H \frac{\partial^2 W_E}{\partial H^2} - \frac{i\hbar}{2} \frac{\partial W_E}{\partial \phi} - \frac{\hbar^2}{16H} \frac{\partial^2 W_E}{\partial \phi^2} \\ &= 0. \end{aligned} \tag{6.3}$$

Since the function W_E is real we see that

$$\frac{i\hbar}{2} \frac{\partial W_E}{\partial \phi} = 0.$$

Thus, W_E depends only on H and may contain \hbar in the denominator. It is possible that for some solutions of (6.3) the series $\sigma^{-1}(W_E)$ is not well defined. Such functions are not admissible because for them the product $\sigma^{-1}(H) \circ \sigma^{-1}(W_E)$ does not exist. The only acceptable solutions of (6.3) are Wigner functions

$$W_{E_n}(H) = \frac{1}{\pi \hbar} (-1)^n \exp\left(\frac{-2H}{\hbar}\right) L_n\left(\frac{4H}{\hbar}\right),$$

where L_n is a Laguerre polynomial, and the energy E is quantized

$$E_n = \hbar(n + \frac{1}{2}), \quad n = 0, 1, 2, \dots$$

This result is well known from traditional quantum mechanics. It can be also obtained in terms of the Moyal product (1.4) [12].

6.3. *Two-dimensional phase space with constant curvature tensor*

Let us consider the phase space (\mathbb{R}^2, ω) . Topologically it is homeomorphic to \mathbb{R}^2 . We cover it with an atlas containing only one chart $(\mathbb{R}^{2n}, \varrho)$. In this chart the symplectic form is, as usual, $\omega = dp \wedge dq$, where q denotes the spatial coordinate and p the momentum conjugate to q . We assume that in the chart (\mathbb{R}^2, ϱ) the connection 1-form is

$$\Gamma = \frac{1}{2} p X^1 X^1 dq. \tag{6.4}$$

The connection (6.4) is well defined globally since we cover the symplectic manifold (\mathbb{R}^2, ω) with only one chart. The curvature 2-form

$$R = -\frac{1}{2} X^1 X^1 dq \wedge dp.$$

Hence, in the chart (\mathbb{R}^2, ϱ) the curvature tensor has only one nonvanishing component

$$R_{1112} = -R_{1121} = -\frac{1}{4}.$$

The Ricci tensor is

$$K_{11} = \frac{1}{4}.$$

Since the connection 1-form is defined by the expression (6.4), we are able to build the abelian connection. Let us compute the series r defined by (4.36). The first term in the recurrent expression (4.36) is

$$\delta^{-1} R = \frac{1}{8} X^1 X^1 X^2 dq - \frac{1}{8} X^1 X^1 X^1 dp.$$

After some computations (for details see [86]) we obtain that the abelian connection is

$$\begin{aligned} \tilde{\Gamma} = & -X^1 dp + X^2 dq + \frac{1}{2} p X^1 X^1 dq - \frac{1}{8} X^1 X^1 X^1 dp + \frac{1}{8} X^1 X^1 X^2 dq \\ & + \frac{1}{128} X^1 X^1 X^1 X^1 X^1 dp - \frac{1}{128} X^1 X^1 X^1 X^1 X^2 dq \\ & - \frac{1}{1024} X^1 X^1 X^1 X^1 X^1 X^1 X^1 dp + \frac{1}{1024} X^1 X^1 X^1 X^1 X^1 X^1 X^2 dq - \dots \end{aligned} \tag{6.5}$$

Note that the abelian connection is an infinite series, the Planck constant \hbar does not appear in any term of the series, and in the series r (4.26) only two kind of terms are present:

$$(-1)^{m+1} a_{2m+1} (X^1)^{2m} X^2 dq$$

and

$$(-1)^m a_{2m+1} (X^1)^{2m+1} dp, \quad m \in N - \{0\},$$

where a_{2m+1} is related to the so called Catalan numbers and is given by

$$a_{2m+1} = \frac{2}{m} \binom{2m-2}{m-1} \frac{1}{16^m}, \quad m \geq 1. \tag{6.6}$$

The expression (6.5) allows us to write eigenvalue equations for observables. Let us start with constructing the explicit form of the eigenvalue equation for momentum p . The general form of this equation is

$$p * W_{\mathbf{p}}(q, p) = \mathbf{p} \cdot W_{\mathbf{p}}(q, p), \tag{6.7}$$

where \mathbf{p} denotes the eigenvalue of p and $W_{\mathbf{p}}(q, p)$ is the Wigner function associated to the eigenvalue \mathbf{p} . We find that

$$\begin{aligned} \sigma^{-1}(p) &= p + X^2 + \frac{1}{2}pX^1X^1 + \frac{1}{8}X^1X^1X^2 - \frac{1}{128}X^1X^1X^1X^1X^2 \\ &\quad + \frac{1}{1024}X^1X^1X^1X^1X^1X^2 - \dots \end{aligned}$$

Every coefficient b_{2m+1} appearing in the term $b_{2m+1} \underbrace{X^1 \dots X^1}_{2m} X^2$, for $m > 0$, can be expressed in the form

$$b_{2m+1} = (-1)^{m+1} a_{2m+1},$$

where a_{2m+1} is defined by (6.6).

Much more complicated is to find the general formula of the series $\sigma^{-1}(W_{\mathbf{p}})$ representing the Wigner function $W_{\mathbf{p}}(q, p)$. After long considerations we conclude that the function $W_{\mathbf{p}}(q, p)$ depends only on p and that the eigenvalue equation for $W_{\mathbf{p}}(p)$ is the infinite differential equation

$$p \cdot \left(W_{\mathbf{p}}(p) - \frac{1}{8}\hbar^2 \frac{d^2 W_{\mathbf{p}}(p)}{dp^2} - \frac{1}{128}\hbar^4 \frac{d^4 W_{\mathbf{p}}(p)}{dp^4} - \dots \right) = \mathbf{p} \cdot W_{\mathbf{p}}(p).$$

The eigenvalue equation for momentum p is a differential equation of infinite degree. There is no general method for solving such equations. Therefore, we decided to look for the solution of the eigenvalue equation of p^2 . As $p \cdot p = p * p$ the Wigner function fulfilling Eq. (6.7) satisfies also the relation

$$p^2 * W_{\mathbf{p}}(p) = \mathbf{p}^2 W_{\mathbf{p}}(p).$$

We can see that the above relation is the modified Bessel equation

$$\frac{1}{4}\hbar^2 p^2 \frac{d^2 W_{\mathbf{p}}(p)}{dp^2} + \frac{1}{4}\hbar^2 p \frac{dW_{\mathbf{p}}(p)}{dp} + (\mathbf{p}^2 - p^2)W_{\mathbf{p}}(p) = 0. \tag{6.8}$$

The general solution of (6.8) is a linear combination of the following form

$$W_{\mathbf{p}}(p) = A \cdot I_{2i\mathbf{p}/\hbar} \left(\frac{2p}{\hbar} \right) + B \cdot K_{2i\mathbf{p}/\hbar} \left(\frac{2p}{\hbar} \right),$$

with $I_{2i\mathbf{p}/\hbar}(2p/\hbar)$ the modified Bessel function with complex parameter $2i\mathbf{p}/\hbar$ and $K_{2i\mathbf{p}/\hbar}(2p/\hbar)$ the modified Bessel function of second kind with parameter $2i\mathbf{p}/\hbar$. This solution is defined for arguments $(2p/\hbar) > 0$ (see [97]). Note that the deformation parameter \hbar appears in the denominator of the argument. We must be very careful because this fact may cause the nonexistence of the series $\sigma^{-1}(W_{\mathbf{p}}(p))$.

The function $A \cdot I_{2i\mathbf{p}/\hbar}(2p/\hbar)$ is complex. Its real part grows up to infinity for $x \rightarrow \infty$ and, hence, it is not normalizable. The imaginary part of $I_{2i\mathbf{p}/\hbar}(2p/\hbar)$ is proportional to $K_{2i\mathbf{p}/\hbar}(2p/\hbar)$. Thus, the only physically admissible solution of (6.8) is

$$W_{\mathbf{p}}(p) = B \cdot K_{2i\mathbf{p}/\hbar} \left(\frac{2p}{\hbar} \right).$$

However, solutions defined on the whole R are required. It is impossible to define the solution of the equation (6.8) on the whole axis. The problem is that the modified Bessel function of second kind is not defined for the argument value $p = 0$. Moreover, $\lim_{p \rightarrow 0^+} K_{2i\mathbf{p}/\hbar} \left(\frac{2p}{\hbar} \right)$ does not exist.

Let us assume that the Wigner function $W_{\mathbf{p}}(p)$ is a generalized function over the Schwartz space $\mathcal{S}(p)$ of test smooth functions tending to 0 when $p \rightarrow \pm\infty$ faster than the inverse of any polynomial.

We define the Wigner function $W_{\mathbf{p}}(p)$ as:

1. $\mathbf{p} < 0$

$$W_{\mathbf{p}}(p) = \begin{cases} \frac{4}{\pi\hbar} \cosh \frac{\mathbf{p}\pi}{\hbar} K_{2i\mathbf{p}/\hbar} \left(\frac{-2p}{\hbar} \right) & \text{for } p < 0, \\ 0 & \text{for } p \geq 0. \end{cases}$$

2. $\mathbf{p} > 0$

$$W_{\mathbf{p}}(p) = \begin{cases} 0 & \text{for } p \leq 0, \\ \frac{4}{\pi\hbar} \cosh \frac{\mathbf{p}\pi}{\hbar} K_{2i\mathbf{p}/\hbar} \left(\frac{2p}{\hbar} \right) & \text{for } p > 0. \end{cases}$$

3. $\mathbf{p} = 0$

$$W_0(p) = \begin{cases} \frac{2}{\pi\hbar} K_0 \left(\frac{-2p}{\hbar} \right) & \text{for } p < 0, \\ \frac{2}{\pi\hbar} K_0 \left(\frac{2p}{\hbar} \right) & \text{for } p > 0. \end{cases}$$

The example analyzed before is a particular choice of G in the 2-form of curvature

$$R = \pm G^2 X^1 X^1 dq \wedge dp, \tag{6.9}$$

where G is some positive constant. The solutions of the equation (6.9) are divided in two classes for $R = G^2 X^1 X^1 dq \wedge dp$ and $R = -G^2 X^1 X^1 dq \wedge dp$, respectively. For $G \rightarrow 0$ both of them tend to $\delta(p - \mathbf{p})$.

The eigenvalue equation for position q takes the form

$$q * W_{\mathbf{q}}(q, p) = \mathbf{q} W_{\mathbf{q}}(q, p),$$

where \mathbf{q} denotes the eigenvalue of the position q . It can be separated in two parts: the real part

$$q \cdot W_{\mathbf{q}}(q, p) = \mathbf{q} W_{\mathbf{q}}(q, p) \tag{6.10}$$

and the imaginary part

$$\frac{1}{2} \hbar \frac{\partial W_{\mathbf{q}}(q, p)}{\partial p} + \frac{1}{96} \hbar^3 \frac{\partial^3 W_{\mathbf{q}}(q, p)}{\partial p^3} + \dots = 0. \tag{6.11}$$

Let us start with Eq. (6.11). Multiplying it by $2/\hbar$, introducing a new variable $z = \sqrt{2}p/\hbar$ and defining $w_{\mathbf{q}}(q, z) = \partial W_{\mathbf{q}}(q, z)/\partial z$, we obtain the formula

$$w_{\mathbf{q}}(q, p) + \frac{1}{24} \frac{\partial^2 w_{\mathbf{q}}(q, p)}{\partial z^2} + \dots = 0. \tag{6.12}$$

This is, in fact, a homogeneous linear differential equation of infinite degree. Its solution neither depends on the parameter \mathbf{q} nor the factor G . Since the $\lim_{G \rightarrow 0^+} w_{\mathbf{q}}(q, z)$ must be 0 the only admissible solution of (6.12) is $w_{\mathbf{q}}(q, z) = 0$ for every \mathbf{q} . We see that the Wigner eigenfunction $W_{\mathbf{q}}(q, z)$ depends only on q . From (6.10) we immediately obtain that

$$W_{\mathbf{q}}(q, z) = \delta(q - \mathbf{q}) \quad \forall \mathbf{q}.$$

Similar considerations can be done for any arbitrary curvature 2-form R of the form (6.9). The final result will be the same, i.e. the eigenfunctions $W_{\mathbf{q}}(q)$ depend only on q and have the same form like in the case of the flat space \mathbb{R}^{2n} with $\Gamma = 0$.

It is well known [91,92] that on a two-dimensional symplectic manifold all the $*$ -products are equivalent. Thus, for every two products $*_1$ and $*_2$ the relation (5.4) holds. It means that the $*$ -product considered in this example is equivalent to the Moyal product (1.4). That is true but we have not algorithm which enables us to construct the operator \hat{T} from the equation (5.4). We cannot transform Wigner eigenfunctions and eigenvalues solutions of the eigenvalue equation

$$F(p, q) *_1 W_{F1}(p, q) = F_1 \cdot W_{F1}(p, q)$$

into eigenvalues and eigenfunctions of

$$F(p, q) *_2 W_{F2}(p, q) = F_2 \cdot W_{F2}(p, q)$$

though we are aware of the existence of some relation between them.

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